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JOURNAL OF ALGEBRA 105, 149–158 (1987)

A New Relation among Cartan Matrix and Coxeter Matrix

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Communicated by Walter Feit

Received February 4, 1985

INTRODUCTION

Let R be a root system of type A_l (l: odd), D_l or one of E_6, E_7, and E_8. The purpose of this note is to show a numerical equality:

$$pqr/d = abc/h.$$

Here pqr/d is a numerical invariant concerning about the Cartan matrix of R defined in Section 1 and abc/h is a numerical invariant concerning about a Coxeter transformation of R defined in Section 2.

We give a proof of the equality in Section 4 using the self-intersection number of a Weil divisor on a rational surface. The argument is a version of that for the strange duality on the 14 exceptional unimodular singularities of Arnold due to Pinkham (cf. [1, 6]). It might be quite interesting to find a purely arithmetic proof of the equality.

The author would like to express his glatitude to his colleages I. Naruki and M. Tomari for their interest and helpfull discussions and also to K. Ueno, who has noticed the author on some works on anti-canonical divisors.

1. CARTAN MATRIX

Let Γ be the Dynkin diagram for a finite root system R without a multiple bond. We understand Γ as a diagram with three branches of length p, q, and r [2].

For the case of type A_l, we interpretate Γ to be a diagram branching at the middle point. Thus we disclude the case when l is even from our consideration.

0021-8693/87 $3.00

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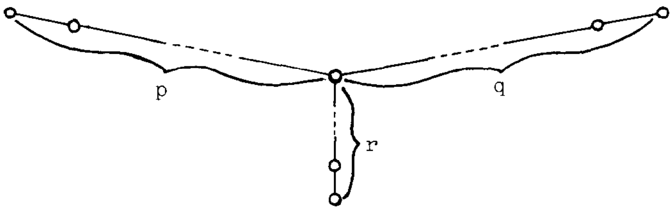


We denote by d the determinant of the Cartan matrix of R. As one calculates easily,

$$d = -pqr + pq + qr + rp.$$

Thus we obtain Table I for p, q, r.

2. Coxeter Transformation

Let h be the order of a Coxeter transformation (= the Coxeter number) for R. Exponents for R are defined as integers $0 < m_1 < \cdots < m_l < h$ such that $\exp((m_i/h)\, 2\pi\sqrt{-1})$ $(i = 1,..., l)$ form the set of eigenvalues of a Coxeter transformation [2].

For a suitable positive integers a, b, and c, we have the following generating function for the set of exponents (cf. Note 1),

$$T^{m_1} + T^{m_2} + \cdots + T^{m_l} = T^{-h} \frac{(T^h - T^a)(T^h - T^b)(T^h - T^c)}{(T^a - 1)(T^b - 1)(T^c - 1)}.$$

In this note, we shall call (a, b, c) the weights for the root system R.

One calculates directly the Table of weights, Table II, using the data of exponents in [2].

Note 1. This is an immediate consequence of a general formula for the set of exponents for weighted homogeneous hypersurface singularities,

TABLE I

Type	p	q	r	d	pqr/d
A_l	$\frac{l+1}{2}$	$\frac{l+1}{2}$	1	$l+1$	$\frac{l+1}{4}$
D_l	2	2	$l-2$	4	$l-2$
E_6	3	3	2	3	6
E_7	4	3	2	2	12
E_8	5	3	2	1	30

TABLE II

Type	a	b	c	h	abc/h
A_l	1	$\frac{l+1}{2}$	$\frac{l+1}{2}$	$l+1$	$\frac{l+1}{4}$
D_l	2	$l-2$	$l-1$	$2(l-1)$	$l-2$
E_6	3	4	6	2	6
E_7	4	6	9	18	12
E_8	6	10	15	30	30

where we modify the definition of weights and exponents suitably. (cf. [4, Table; 8, (3.7.1)]).

We identify the rational numbers a/h, b/h, c/h with the weights r_x, r_y, r_z of the coordinates x, y, z for the weighted homogeneous polynomial $f(x, y, z)$ of degree 1, which defines the rational double point corresponding to the root system.

Note 2. The weights and the Coxeter number are related by the following equality: $a+b+c=h+1$.

3. Comparison

By compairing tables in Sections 1 and 2, we obtain:

THEOREM. *Let R be a root system of type A_l (l: odd), D_l, or one of E_6, E_7, and E_8. Then the equality*:

$$d/pqr = h/abc$$

holds.

Note. The left-hand side of the equality can be interpreted as the spherical volume of the Schwarz triangle on $\mathbb{P}^1(\mathbb{C})$ for the regular polyhedral group in $PGL_2(\mathbb{C})$ which corresponds to the type of the root system.

The same type of the equality holds for "indefinite root systems," where the left-hand side of the equality is interpreted as the volume of the fundamental domain in the upper half plane for the corresponding Fuchsian group of the first kind (cf. [7, (5.6.3), (5.7.3)]).

4. A Proof of the Theorem

In this pararaph, we shall construct a singular rational surface $\bar{X}$ with a Weil divisor C in it. The self-intersection number C^2 is calculated by two different ways: one method gives the values h/abc and the other d/pqr (cf. (4.5)). This gives an interpretation of the *assertion* of Section 3. The rational surface was once used by Brieskorn for a study of resolution for a mapping of type, E_8 (cf. [3]).

(4.1.) Let (a, b, c) be weights and let $h = a + b + c - 1$ be the Coxeter number for a root system R as in Table II.

There exists a weighted homogeneous polynomial $f(x, y, z)$ of degree h in three variables x, y, and z by giving weights a, b, and c for the variable, such that the hypersurface in $\mathbb{C}^3$,

$$X_0: f(x, y, z) = 0$$

has an isolated singular point at 0, which is known as a rational double point. For instance, A_l: $x^{l+1} + y^2 + z^2$, D_l: $x^2y + y^{l-1} + z^2$, E_6: $x^4 + y^3 + z^2$, E_7: $x^3y + y^3 + z^2$, E_8: $x^5 + y^3 + z^2$ (cf. [4]).

The smooth affine variety in $\mathbb{C}^3$,

$$X: f(x, y, z) = 1$$

will be called the Milnor fiber.

(4.2) A compactification of the Milnor fiber X is given by a compact hypersurface in $\mathbb{P}(a, b, c, 1)$,

$$\bar{X}: f(x, y, z) = w^h,$$

where the weighted projective space $\mathbb{P}(a, b, c, 1)$ is a quotient of $\mathbb{C}^4 - \{0\}$ by the relation $(x, y, z, w) \sim (t^a x, t^b y, t^c z, tw)$ for $t \in \mathbb{C}^*$.

There is a natural embedding,

$$X \subset \bar{X}, \qquad (x, y, z) \mapsto (x : y : z : 1)$$

so that the complement curve,

$$C := \bar{X} - X = \bar{X} \cap \mathbb{P}^2(a, b, c)$$

(here $\mathbb{P}^2(a, b, c)$: $w = 0$.) is a Weil divisor. (For a description of C at a singular point of $\bar{X}$, see *Assertion* 1 in (4.3).) Note that C is isomorphic to a smooth rational curve $\hat{C} := X_0 - \{0\}/\mathbb{C}^*$ in $\mathbb{P}^2(a, b, c)$ defined by $f(x, y, z) = 0$. ($\hat{C}$ is smooth since $X_0 - \{0\}$ is smooth and is rational since x^b/y^a defines a simple or a double covering of $\hat{C}$ over $\mathbb{P}^1$ branching at most at two points.)

(4.3) The singular points of $\bar{X}$, which automatically lie on C, are described as follows.

ASSERTIONS. 1. *For each singular point* $\mathbf{x}$ *of* $\bar{X}$, *there exists a positive integer* $p > 1$, *such that we have a local isomorphism,*

$$\begin{array}{ccc} (\bar{X}, \mathbf{x}) & \simeq & (\mathbb{C}^2, 0)/\mathbb{Z}_p \\ \cup & & \cup \\ (C, \mathbf{x}) & \simeq & (\mathbb{C}, 0)/\mathbb{Z}_p. \end{array}$$

Here the action of $\mathbb{Z}_p$ *on* $\mathbb{C}^2$ *is given by* $(w, v) \mapsto (\zeta w, \zeta v)$ *for* $\zeta \in \sqrt[p]{1} \simeq \mathbb{Z}_p$, $(w, v) \in \mathbb{C}^2$, *and* $\mathbb{C}$ *is a linear subspace of* $\mathbb{C}^2$ *defined by* $w = 0$ *on which* $\zeta \in \mathbb{Z}_p$ *acts by* $(v) \mapsto (\zeta v)$.

2. *The set of the orders p of the cyclic groups for the singularities of* $\bar{X}$ *coincides with the* $\{p, q, r\}$, *where* p, q, r *is given in Table I in Section* 1 (*cf. Note* 1).

Proof. The chart $x \neq 0$ of $\mathbb{P}(a, b, c, 1)$ is given as a quotient of $1 \times \mathbb{C}^3$ by the relation,

$$(1, y, z, w) \sim (1, \zeta^b y, \zeta^c z, \zeta w) \qquad \text{for} \quad \zeta \in \sqrt[a]{1} \simeq \mathbb{Z}_a,$$

so that $\bar{X}$ on that chart is given as a quotient variety of the affine variety in $1 \times \mathbb{C}^3$ defined by,

$$Y: f(1, y, z) - w^h = 0$$

by the action of $\mathbb{Z}_a$.

If a point $\mathbf{x} = (1, y, z, w)$ of Y is a fixed point of the action, then $w = 0$. If $\mathbf{x} = (1, 0, 0, 0) \in Y$, then the isotropy group at the point is $\mathbb{Z}_a$. If $\mathbf{x} = (1, y, 0, 0) \in Y$ for a $y \neq 0$, then the isotropy group at the point is $\sqrt[a]{1} \cap \sqrt[b]{1} = Z_{gcd(a,b)}$. If $\mathbf{x} = (1, 0, z, 0) \in Y$ for a $z \neq 0$, then the isotropy group at the point is $\sqrt[a]{1} \cap \sqrt[c]{1} \simeq \mathbb{Z}_{gcd(a,c)}$.

We may choose $f(1, y, z) - w^h$ and w as a part of local coordinates of $1 \times \mathbb{C}^3$ at a fix point $\mathbf{x}$ of Y, since the rank of

$$\frac{\partial(f - w^h, w)}{\partial(y, z, w)} = \begin{pmatrix} \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z} & -hw^{h-1} \\ 0 & 0 & 1 \end{pmatrix}$$

at $\mathbf{x}$ is equal to two. (Otherwise $\partial f/\partial y$, $\partial f/\partial z$, w, and $f - w^h$ are zero at that point and hence also $\partial f/\partial x$ is zero. This contradicts to the isolatedness of the singular point of X_0.) Recall an obvious relation,

$$(f - w^h)(\zeta(1, y, z, w)) = \zeta^h(f - w^h)(1, y, z, w)$$

for $\zeta \in \mathbb{Z}_a$. Therefore the Jacobian of the axiom of $\zeta \in \mathbb{Z}_a$ at a fixed point $\mathbf{x}$ on Y is calculated as,

$$\frac{\partial(\zeta^b y, \zeta^c z, \zeta w)}{\partial(y, z, w)} \Big/ \frac{\partial((f - w^h) \circ \zeta}{\partial(f - w^h)} = \zeta^{b+c+1-h} = \zeta^{2-a} = \zeta^2.$$

Since ζ acts as a multiplication on one coordinate w of Y at $\mathbf{x}$, it acts as a multiplication on the other local coordinate v of Y at $\mathbf{x}$. This gives the description of Assertion 1.

By a cyclic change of the roll of x, y, z, we obtain all singular points of $\bar{X}$.

2. Due to the description of the above 1, all singular points of $\bar{X}$ are on $C \cap (l_x \cup l_y \cup l_z)$, where l_x, l_y, and l_z are the coordinate axis of $\mathbb{P}^2(a, b, c)$.

By introducing a notation,

$$m(a_1, \ldots, a_k) := \# \left\{ (n_1, \ldots, n_k) \in (\mathbb{Z}_{\geqslant 0})^k : h = \sum_{i=1}^{k} n_i a_i \right\}$$

for $k \in \mathbb{N}$ and putting $a_1 := a$, $a_2 := b$, $a_2 := c$ and $l_1 := l_x$, $l_2 := l_y$, $l_3 := l_z$, we obtain the set A of the orders of isotropy groups at the point $C \cap (l_1 \cup l_2 \cup l_3)$ as follows:

$$A = \bigcup_{\substack{i=1 \\ m(a_i)=0}}^{3} \{a_i\} \cup \bigcup_{\substack{i,j=1 \\ i \neq j}}^{3} \{gcd(a_i, a_j) : m(a_i, a_j) - 1 \text{ times}\}$$

($\because$ Let i, j, k be a permutation of 1, 2, 3. Then one checks easily the following relations:

(i) $l_i \cap l_j \cap l_k = \emptyset$,

(ii) $C \cap l_i \cap l_j \neq \emptyset \Leftrightarrow m(a_k) = 0$,

(iii) $\# C \cap (l_i - (l_j \cup l_k)) = m(a_j, a_k) - 1$.)

(Here in the definition of A, we count the elements of A together with is its multiplicity.)

By deleting the element 1's from A, we obtain the set of orders of groups for the singular points of $\bar{X}$. By a direct calculation of the set A, one proves 2 of the assertion. Q.E.D.

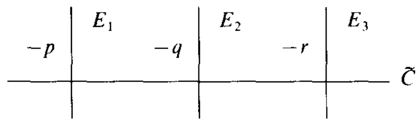
Note 1. To be exact, the statement of Assertion 2 is true up to the elements equal to 1. Therefore one needs to modify the statement properly for the cases of types A_l.

Note 2. As we have seen in the construction of the set A in the proof of the Assertion 2, *the set* $\{p, q, r\}$ *of Table* I *is determined from the set*

$\{a, b, c\}$ *of the Table* II *by purely arithmetic procedure without using any transcendental (topological) means.*

Note 9. The integral multiple $pqrC$ of C is a Cartier divisor in $\bar{X}$.

(**4.4**) Let $\pi: \tilde{X} \to X$ be the minimal resolution of singularities of $\bar{X}$ and let $\tilde{C}$ be a strict transform of C. As a consequence of Assertion 1 of (**4.3**), the exceptional set $E := \pi^{-1}(\mathbf{x})$ for a singular point $\mathbf{x} \in \bar{X}$ is a smooth rational curve with the self-intersection number $E^2 = -p$ (cf. also a proof of Assertion 3 below). The curves $\tilde{C}$ and E intersect at a point transversally. Thus the divisor $\tilde{X} - X$ has the following figure,



where p, q, r is in Table I.

Assertions 3. *The canonical divisor* $K_{\tilde{X}}$ *of* $\tilde{X}$ *is given by*

$$K_{\tilde{X}} = -(2\tilde{C} + E_1 + E_2 + E_3).$$

4. *The self-intersection number of* $\tilde{C}$ *is given by*

$$\tilde{C}^2 = -1.$$

Proof. We construct a meromorphic section ω of the canonical bundle of $\tilde{X}$. First, let us consider a 3-form on $\mathbb{C}^4$

$$\Omega := \frac{(ax\,dy\,dz + by\,dz\,dx + cz\,dx\,dy)\,dw + w\,dx\,dy\,dz}{w^2(f(x, y, z) - w^h)}.$$

Since $\deg(\Omega) = 0$ and

$$\left\langle ax\frac{\partial}{\partial x} + by\frac{\partial}{\partial y} + cz\frac{\partial}{\partial z} + w\frac{\partial}{\partial w}, \Omega \right\rangle = 0,$$

the form Ω induces a meromorphic 3 form on $\mathbb{P}(a, b, c, 1)$, having a simple pole along $\bar{X}$ and a double pole along $\mathbb{P}(a, b, c)$, which we shall denote again by Ω. Let us define a two form $\bar{\omega}$ on $\bar{X}$ by the residue,

$$\bar{\omega} := \mathrm{Res}_{\bar{X}}(\Omega).$$

Since on the affine chart $\mathbb{C}^3 \times 1$ of $\mathbb{P}(a, b, c, 1)$, Ω has an expression $dx\,dy\,dz/(f(x, y, z) - 1)$, $\bar{\omega}$ does not have zeros or poles on the Milnor fiber $X = \bar{X} \cap (\mathbb{C}^3 \times 1)$.

Denote by ω the lifting of $\bar{\omega}$ from $\bar{X}$ to $\tilde{X}$. Let us show that ω has a pole of order 2 along $\tilde{C}$ and simple poles along the exceptional divisors E_i $(i=1, 2, 3)$.

As in the proof of Assertion 1 in (4.3), let Y be the affine variety in $1 \times \mathbb{C}^3(y, z, w)$ defined by the equation $f(1, y, z) - w^h = 0$ on which the group $\mathbb{Z}_a$ acts so that $Y/\mathbb{Z}_a$ is an affine chart of $\bar{X}$. Since $\Omega \mid 1 \times \mathbb{C}^3 = a\, dy\, dz\, dw/w^2(f(1, y, z) - w^h)$, we have $\operatorname{Res}_Y(\Omega) = a\, dz\, dw/w^2\, \partial f/\partial y = -a\, dy\, dw/w^2\, \partial f/\partial z = g(dw\, dv/w^2)$, where (w, v) is the local coordinates of Y in Assertion 1 of (4.3) and g is a unit invariant by the isotropy group.

At a fixed point $\mathbf{x}$ of Y, the quotient map $(Y, \mathbf{x}) \to (Y/\mathbb{Z}_p, \mathbf{x}) \simeq (\bar{X}, \mathbf{x})$ is described by a Veronese map $\mathbb{C}^2 \to \mathbb{C}^{p+1}$, $(w, v) \mapsto (w^p, w^{p-1}v, \ldots, v^p)$, and the resolution of $\mathbb{C}^2/\mathbb{Z}_p$ is obtained by a strict transform of the blowing up of $\mathbb{C}^{n+1}$ at the origin:

$$\begin{array}{ccc} (\tilde{X}, E) & \subset & (\tilde{\mathbb{C}}^{n+1}, \mathbb{P}^n) \\ \downarrow & & \downarrow \\ (\bar{X}, \mathbf{x}) & \subset & (\mathbb{C}^{n+1}, 0). \end{array}$$

Thus one may choose $W := w^h$ and $U := w/v$ as local coordinates for an affine chart of $\tilde{X}$ at $\tilde{C} \cap E$, s.t. $W = 0$ defines the divisor E and $U = 0$ defines the divisor $\tilde{C}$.

Using W and U, ω is expressed locally at the point as

$$\omega = g\,\frac{dw\, dv}{w^2} = -\frac{g}{h}\,\frac{1}{w^h}\left(\frac{v}{w}\right)^2 dw^h\, d\left(\frac{w}{v}\right) = -\frac{g}{h}\,\frac{dW\, dU}{WU^2}.$$

This implies $\omega \in \Gamma(\tilde{X}, \mathcal{O}_{\tilde{X}}^{*}(K + 2\tilde{C} + E_1 + E_2 + E_2))$, which proves Assertion 3.

Proof of Assertion 4. Apply the adjunction formula on $\tilde{X}$ for the curve $\tilde{C}$.

$$2g - 2 = \tilde{C}^2 + K_{\tilde{X}}\tilde{C},$$

where $g =$ genus of $\tilde{C} = 0$. By substituting $K_{\tilde{X}} = -2\tilde{C} - E_1 - E_2 - E_3$ and applying $\tilde{C}E_i = 1$ for $i = 1, 2, 3$, we obtain the result. Q.E.D.

(4.5) This paragraph is the goal of this note, where we calculate the selfintersection C^2 of the Weil divisor C in $\bar{X}$ by two different ways.

(i) *The first method.* According to Mumford [5, II], we first calculate the total transform C' on $\tilde{X}$ for C, taking because of Assertions 1 and 4,

$$C' = \tilde{C} + \frac{1}{p}E_1 + \frac{1}{q}E_2 + \frac{1}{r}E_3.$$

Therefore,

$$C^2 := (C')^2 = -1 + \frac{1}{p} + \frac{1}{q} + \frac{1}{r} = \frac{d}{pqr}.$$

(ii) *The second method.* Let us denote by C_x, C_y, and C_z the curves in $\bar{X}$, defined by $x=0$, $y=0$, and $z=0$, respectively. Since $\lambda x + \mu w^a$, $\lambda y + \mu w^b$, and $\lambda z + \mu w^c$ form linear systems of rational Cartier divisors on $\tilde{X}$, we get numerical equivalences $aC \sim C_x$, $bC \sim C_y$, and $cC \sim C_z$. Therefore,

$$C^2 = \frac{1}{ab} C_x \cdot C_y = \frac{1}{ab}\frac{h}{c} = \frac{h}{abc}.$$

(iii) Comparing (i) and (ii) above, one obtains the equality,

$$\frac{d}{pqr} = \frac{h}{abc}.$$

(4.6) *Some other consequences on* $\tilde{X}$. As consequences of Assertions 1, 2, 3, and 4, let us show some properties of $\tilde{X}$.

ASSERTION 5. (i) *$\tilde{X}$ is a rational surface, whose second Betti number is $l+4$. Here $l :=$ rank $R = p+q+r-2 = (h/a-1)(h/b-1)(h/c-1)$.*

(ii) *The Milnor lattice $H_2(X, \mathbb{Z})$ and the lattice $\mathbb{Z}\tilde{C} \oplus \bigoplus_{i=1}^{3} \mathbb{Z}E_i$ are primitively embedded in $H_2(\tilde{X}, \mathbb{Z})$, so that they are orthogonal complement of each other.*

Proof. Due to Brieskorn [4], the Milnor lattice $H_2(X, \mathbb{Z})$ is isomorphic to the root lattice of the corresponding root system. Particularly it has rank l and the intersection form is negative definite, whose discriminant is equal to $(-1)^l d$.

One calculates easily, $c_2 :=$ Euler number of $\tilde{X} =$ Euler number of $X +$ Euler number of $(\tilde{X} - X) = l + 6$ and $c_1^2 = K_{\tilde{X}}^2 = (2\tilde{C} + E_1 + E_2 + E_3)^2 = 8 - p - q - r$. Therefore $\chi(\mathcal{O}_{\tilde{X}}) = \frac{1}{12}(c_1^2 + c_2) = 1$. On the other hand, $p_m = h^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(mK_{\tilde{X}})) = 0$ for $m \geqslant 1$, since $-K_{\tilde{X}}$ is effective. Thus applying the Castelnuovo's criterium, we have shown (i).

The intersection matrix of the lattice $L := \mathbb{Z}\tilde{C} \oplus \bigoplus_{i=1}^{3} \mathbb{Z}E_i$ is

$$\begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -p & 0 & 0 \\ 1 & 0 & -q & 0 \\ 1 & 0 & 0 & -r \end{pmatrix}.$$

Therefore it defines an indefinite form of signature $(1, 3)$, whose discriminant is equal to $pqr - pq - qr - rp = -d$. Particularly it is non-degenerate so that the lattice L is embedded in $H_2(\tilde{X}, \mathbb{Z})$. Since X and $\tilde{X} - X$ do not intersect, $H_2(X, \mathbb{Z})$ and L are orthogonal to each other. Since $H_2(\tilde{X}, \mathbb{Z})$ is unimodular ($\because$ Poincare duality), the coincidence of the discriminants of the two sublattices (up to sign) proves the (ii). Q.E.D.

Notes added in proof. 1. In a discussion I. G. MacDonald has notified the author that the characteristic function $T^{m_1} + \cdots + T^{m_l}$ decomposes into cyclotomic polynomials for all finite Coxeter groups, whose meaning is not yet clear.

2. I. V. Dolachev has notice to the author that the equality $d/pqr = h/abc$ can be obtained by studing certain intersection numbers on the resolution of the singularities, which is a dual procedure to the compactification as done in this note (cf. P. Orlik and P. Wagreich, Isolated singularities with $\mathbb{C}^*$-actions, *Ann. Math.* **93** (1971), 205–228).

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